JOURNAL OF APPROXIMATION THEORY 41, 179–184 (1984)

Exposing Functionals on $C(Q)$

S. SRINIVASAN AND SUNDARAM M. A. SASTRY*

School of Mathematics, Madurai Kamaraj University, Madurai–625021, India

Communicated by Oved Shisha

Received July 24, 1978; revised October 3, 1979

1. INTRODUCTION

In this paper, we investigate questions of existence and density of exposing functionals on $C(Q)$, the Banach space of real-valued continuous functions on a compact Hausdorff space Q under the supremum norm. We show that the exposing functionals on $C(Q)$ are dense in the unit sphere of the dual of $C(Q)$ if, and only if, Q is totally disconnected, and admits a strictly positive measure. The above is true if, and only if, the Chebyshev hyperspaces of $C(Q)$ are dense in the set of all hyperspaces of $C(Q)$, provided with the metric θ', introduced by Brown [3]. As a corollary, we show that $L_\infty(T, \Sigma, \mu)$ has a dense set of Chebyshev hyperspaces if μ is σ-finite.

2. NOTATIONS AND TERMINOLOGY

All Banach spaces are over the real field. If X is a Banach Space, X^* stands for the dual space,

$$B(X) = \{x \in X : \|x\| \leqslant 1\} \qquad \text{and} \qquad S(X) = \{x \in X : \|x\| = 1\}.$$

A hyperspace of a Banach space is a closed linear subspace of codimension one. The set of all hyperspaces will be denoted by $\mathbf{H}$. For $M_1, M_2 \in \mathbf{H}$,

$$\theta(M_1, M_2) = \max[\sup_{m_1 \in S(M_1)} \{\inf_{m_2 \in M_2} (\|m_1 - m_2\|\}, \sup_{m_2 \in S(M_2)} \{\inf_{m_1 \in M_1} (\|m_1 - m_2\|)\}].$$

For $M_1, M_2 \in \mathbf{H}$, $\theta'(M_1, M_2)$ is the Hausdorff distance between $S(M_1)$ and $S(M_2)$. It is easy to see that $\{(M, N) \in \mathbf{H} \times \mathbf{H} : \theta(M, N) < \varepsilon\}_{\varepsilon > 0}$ is a base for a uniformity on $\mathbf{H}$, and the uniform topology on $\mathbf{H}$ is the same as the metric

* This research was supported in part by Project No. 10331 of the University Grants Commission, India.

0021-9045/84 $3.00

topology of $(\mathbf{H}, \theta')$. In fact, we have the following inequalities, the proof of which can be found in [3, 7]: $\frac{1}{2}\theta' \leqslant \theta \leqslant \theta'$.

Let X be a Banach space. An element $f \in S(X^*)$ is said to be an exposing functional on X if there is $x \in S(X)$ such that $f(x) = 1$ and $f(y) < 1$ for every $x \neq y \in B(X)$. In this case, x is said to be an exposed point of $B(X)$ (exposed by f). It is easy to see that every exposed point is an extreme point. Q shall stand for a compact Hausdorff space, and $G(Q)$ for the Banach Space of all real-valued cntinuous functions on Q under the supremum norm.

A measure μ on Q is a regular bounded Borel measure on Q. A measure μ on Q is said to be strictly positive if $\mu(U) > 0$ for every open subset U of Q. Q is said to have property P if Q admits a strictly positive measure. The set of all of compact Hausdorff spaces having property P shall be denoted by $\mathbf{P}$.

3

The following theorem establishes a bijection between exposing functionals and Chebyshev hyperspaces.

THEOREM 3.1. *$f \in S(X^*)$ is an exposing functional if, and only if, $M = \operatorname{Ker} f$ is a Chebyshev hyperspace.*

We obtain a relation between the distance between two elements of $S(X^*)$ and the θ'-distance between their kernels.

THEOREM 3.2. *Let $f_1, f_2 \in S(X^*)$, and let $M_i = \operatorname{Ker} f_i$, $i = 1, 2$. Then $\theta'(M_1, M_2) \leqslant 2 \|f_1 - f_2\|$.*

Proof. Let $m_i \in S(M_1)$. Then

$$\inf_{m_2 \in M_2} (\|m_1 - m_2\|) = d(m_1, M_2) = |f_2(m_1)| = |(f_2 - f_1)(m_1)| \leqslant \|f_2 - f_1\|.$$

Hence,

$$\operatorname{Sup}_{m_1 \in S(M_1)} \{ \inf_{m_2 \in M_2} (\|m_1 - m_2\|)\} \leqslant \|f_2 - f_1\|.$$

Similarly

$$\operatorname{Sup}_{m_2 \in S(M_2)} \{ \inf_{m_1 \in M_1} (\|m_1 - m_2\|)\} \leqslant \|f_2 - f_1\|.$$

Consequently, $\theta(M_1, M_2) \leqslant \|f_2 - f_1\|$, and so $\theta'(M_1, M_2) \leqslant \|f_2 - f_1\|$.

COROLLARY. *If the exposing functionals are dense in $S(X^*)$, then the Chebyshev hyperspaces are dense in $(\mathbf{H}, \theta')$.*

The converse of the above corollary is true. First, we need the following lemma; the proof can be found in [5].

LEMMA. *Let $f, g \in B(X^*)$, and $\varepsilon > 0$. Let $|g(x)| < \varepsilon/2$ whenever $\|x\| \leqslant 1$ and $f(x) = 0$. Then, either $\|f - g\| < \varepsilon$ or $\|f + g\| < \varepsilon$.*

THEOREM 3.3. *Let $f_1, f_2 \in S(X^*)$, let $M_i = \ker f_i$, $i = 1, 2$, and let $\theta'(M_1, M_2) < \varepsilon$. Then, either $\|f - g\| < \varepsilon$, or $\|f + g\| < \varepsilon$.*

Proof. Let $m_1 \in M_1$, $\|m_1\| \leqslant 1$, and $n = m_1/\|m_1\|$. Then

$$|f_2(m_1)| \leqslant |f_2(n)| = d(n, M_2) = \inf_{m_2 \in M_2} (\|n - m_2\|) \leqslant \theta(M_1, M_2)$$

$$\leqslant \theta'(M_1, M_2) < \varepsilon.$$

Hence, by the above lemma, either $\|f_1 - f_2\| < \varepsilon$ or $\|f_1 + f_2\| < \varepsilon$.

COROLLARY. *If the Chebyshev hyperspaces of X are dense in $(\mathbf{H}, \theta')$, then the exposing functionals are dense in $S(X^*)$.*

We apply the above result to $C(Q)$. The dual of $C(Q)$ is $M(Q)$, the Banach space of all regular, bounded Borel measures on Q with the total variation norm.

THEOREM 3.4. *$\mu \in S(M(Q))$ is an exposing measure if, and only if, (i) $S(\mu) = Q$, and*

(ii) $S(\mu^+) \cap S(\mu^-) = \varnothing$,

where $S(\mu)$ stands for the support of μ.

For the proof, refer to [7].

Remarks. (1) if μ is an exposing measure, then μ exposes

$$f \in C(Q),$$

where

$$f(q) = \begin{cases} 1 & \text{if} \quad q \in S(\mu^+) \\ -1 & \text{if} \quad q \in S(\mu^-). \end{cases}$$

(2) If μ is an exposing measure, then $S(|\mu|) = Q$; that is, $|\mu|$ is a strictly positive measure on Q. Thus $Q \in \mathbf{P}$.

The above remark implies, because of Theorem 3.1, that $C(Q)$ has a Chebyshev hyperspace if, and only if, $Q \in \mathbf{P}$. It is therefore important to know when $Q \in \mathbf{P}$.

Separable spaces clearly have property P. On the other hand $\mathbf{P}$ contains many nonseparable spaces since it is closed under products and also contains all compact groups. Some other results on $\mathbf{P}$ are found in [4, 6].

We know that every exposed point is an extreme point. If $Q \in P$, then we have the following theorem.

THEOREM 3.5. *If $Q \in \mathbf{P}$, then every extreme point of $B(C(Q))$ is an exposed point.*

4

We now consider conditions on Q which guarantee a dense set of Chebyshev hyperspaces of $C(Q)$—or, equivalently, because of the corollary to Theorem 3.3, conditions on Q which ensures a dense set of exposing functionals in $M(Q)$. We show that this happens if, and only if, $Q \in \mathbf{P}$ and is totally disconnected.

We first need the following theorem.

THEOREM 4.1. *Let X be a Banach space. If the exposing functionals are dense in $S(X^*)$, then the convex hull of the exposed points of $B(X)$ is dense in $B(X)$.*

Proof. Let K be the closed convex hull of the exposed points of $B(X)$. Suppose $K \subsetneq B(X)$. Let $x \in B(X) \backslash K$.

By the Hahn-Banach theorem, there is an $f \in S(X^*)$ and $\alpha > 0$ such that $f(x) > \alpha$ and $f(y) \leqslant \alpha$ for all $y \in K$. Since the exposing functionals are dense in $S(X^*)$, there is an exposing functional of $g \in S(X^*)$ such that $\|f - g\| < 1 - \alpha$. Let g expose $y_0 \in S(X)$. Since $y_0 \in K$, $f(y_0) \leqslant \alpha$. But $(g - f)(y_0) = g(y_0) - f(y_0) = 1 - f(y_0) \geqslant 1 - \alpha$, which implies $\|g - f\| \geqslant 1 - \alpha$, which is a contradiction.

THEOREM 4.2. *The exposing measures are dense in $S(M(Q))$ if, and only if, $Q \in \mathbf{P}$ and Q is totally disconnected.*

Proof. Suppose the exposing measures are dense in $S(M(Q))$. From the remark following Theorem 3.4, $Q \in \mathbf{P}$. By Theorem 4.1, $B(C(Q))$ is the closed convex hull of its extreme points. This implies that Q is totally disconnected [1].

Let us now assume that Q is totally disconnected and $Q \in \mathbf{P}$. By Bishop Phelp's theorem [2], we know that the set of measures in $S(M(Q))$ which attain their norms is dense in $S(M(Q))$. Thus it suffices to show that the set of exposing measures is dense in the set of norm-attaining measures.

Let μ be a norm-attaining measure in $S(M(Q))$, and let $0 < \varepsilon < 1$. By

Phelp's theorem [4], $S(\mu^+) \cap S(\mu^-) = \varnothing$, and μ attains its norm at $f \in s(C(Q))$, where $f = 1$ on $S(\mu^+)$ and $f = -1$ on $S(\mu^-)$.

Since Q is totally disconnected, there is a clopen set V in Q such that $S(\mu^+) \subset V$ and $V \cap S(\mu^-) = \varnothing$. Since $Q \in \mathbf{P}$, there is a strictly positive measure λ on Q with $\lambda(Q) = 1$. Let ν be the Borel measure defined on an arbitrary Borel subset K of Q by

$$\nu(K) = \left(1 - \frac{\varepsilon}{2}\right) \mu(K \cap S(\mu)) + \frac{\varepsilon/2}{\lambda(Q \setminus S(\mu))} \left|\lambda(K \cap (V \setminus S(\mu))) - \lambda(K \setminus (S(\mu) \cup V))\right|.$$

Then, $\|\nu\| = 1$, $S(\nu^+) = V$, $S(\nu^-) = Q \setminus V$ and $\|\mu - \nu\| < \varepsilon$. Hence, by Theorem 3.3, ν is an exposing measure at a distance less than ε from μ.

5

Let (T, Σ, μ) be a σ-finite measure space. $L_\infty(T, \Sigma, \mu)$ is isometrically isomorphic to $C(Q)$ for a totally disconnected compact Hausdorff space. This fact, together with the above results, yields the following theorem.

THEOREM 5.1. *Let (T, Σ, μ) be a σ-finite measure space. Then $L_\infty(T, \Sigma, \mu)$ has a dense set of Chebyshev hyperspaces.*

Proof. Since μ is σ-finite, $L_\infty(\mu)$ is the dual of $L_1(\mu)$. Again, since μ is σ-finite, there is an $f \in L_1(\mu)$ such that $\|f\|_1 = 1$ and $f > 0$ a.e. (μ). This f, viewed as a bounded, linear functional on $L_\infty(\mu)$, is an exposing functional, and exposes the constant function 1.

$L_\infty(\mu)$ is isometrically isomorphic to $C(Q)$ for a totally disconnected, compact Hausdorff space Q. This $C(Q)$ has an exposing functional, and hence $Q \in \mathbf{P}$. Thus, by Theorem 4.2, $C(Q)$, and consequently, $L_\infty(T, \Sigma, \mu)$ has a dense set of Chebyshev hyperspaces.

COROLLARY. *l_∞ has a dense set of Chebyshev hyperspaces.*

The corollary can be seen to be true, also, by observing that $l_\infty = C(\beta N)$, and βN is a totally disconnected, separable, Compact Hausdorff space.

ACKNOWLEDGMENT

The authors would like to acknowledge suggestions for improvement of the exposition made by the referee.

References

1. W. G. Bade, "The Banach Space $C(S)$," Aarhus University Mathematatisk Institut.
2. E. Bishop and R. R. Phelps, A proof that every Banach space is subreflexive, *Bull. Amer. Math. Soc.* **67** (1961), 97–98.
3. A. L. Brown, "Abstract Approximation Theory," *Matscience*, Madras, India.
4. R. R. Phelps, Chebyshev subspaces of finite codimension in $C(X)$, *Pacific J. Math.* **13** (1963), 647–655.
5. R. R. Phelps, A representation theorem for bounded convex sets, *Proc. Amer. Math. Soc.* **11** (1960), 976–982.
6. H. P. Rosenthal, On injective Banach spaces and the spaces $L_{\infty}(u)$ for finite measure, *Acta Math.* **124** (1970), 205–248.
7. I. Singer, "Best approximation in Normed Linear Spaces by elements of Linear Subspaces," Band 17, Springer Verlag, Berlin/Heidelberg/New York.